2 Introduction to Quantum Codes (Week 2)

In this lecture, we will construct our first quantum error correcting code. We will borrow a lot of terminology and conceptual ideas from the classical error-correction literature. Therefore, we start by reviewing a basic classical error correcting code, known as repetition code.

2.1 Repetition Code

In classical world, our data consists of bits $b \in \{0, 1\}$. Suppose we wish to send a bit from one location to another through a noisy classical channel. The effect of noise in the channel is to flip the bit being transmitted with probability $p > 0$. To deal with this noise, we encode bit b by adding redundant bits.

Definition 2.1 (Repetition Code)**.** *Repetition code encodes $b \in \{0, 1\}$ into 3 bits.*

$$0 \to 000$$
$$1 \to 111$$

The bit strings 000 and 111 are referred to as *codewords*, and also sometimes referred to as the *logical 0* and *logical 1*, since they play the role of 0 and 1 respectively.

We now send all three bits 000 through the channel. Suppose our codeword experiences a single bit flip error, and the output of the channel is 010:

$$000 \xrightarrow{\text{noise}} 010$$

We can correct this bit flip by looking at the majority of the three bits. Since, only one bit has been flipped, this will give us the correct value for the three bits:

$$000 \xrightarrow{\text{noise}} 010 \xrightarrow{\text{majority}} 000$$

On the other hand, if there were 2 or more bit flips, this scheme would fail

$$000 \xrightarrow{\text{lots of noise}} 011 \xrightarrow{\text{majority}} 111 \neq 000$$

Hence, the 3-bit repetition code allows us to correct one bit flip error, and fails to correct 2 or more bit flips. This means we could use repetition code to get reliability in channels where error probability p is small. These channels mostly see at most 1 bit flip error, since the probability of seeing 2 or more bit flip errors scales as $O(p^2)$.

2.2 Quantum Noise

Classically, bit-flip is the only possible type of noise. But a quantum state is susceptible to many more types of noise. Let's look at few examples of quantum error.

Definition 2.2 (Bit flip error). *A bit flip error is described by the unitary*

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

which acts on the computational basis as

$$X|0\rangle = |1\rangle$$
$$X|1\rangle = |0\rangle$$

Definition 2.3 (Phase flip error). *A phase flip error is described by the unitary*

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

which acts on the computational basis as

$$Z|0\rangle = |0\rangle$$
$$Z|1\rangle = -|1\rangle$$

Observation 2.4. *Phase flip error acts like a bit flip error in the Hadamard basis since $Z = HXH$. We check by direct computation.*

$$Z|+\rangle = Z\frac{|0\rangle + |1\rangle}{\sqrt{2}} = \frac{|0\rangle - |1\rangle}{\sqrt{2}} = |-\rangle$$
$$Z|-\rangle = Z\frac{|0\rangle - |1\rangle}{\sqrt{2}} = \frac{|0\rangle + |1\rangle}{\sqrt{2}} = |+\rangle$$

In today's lecture, we will consider a simiplified model of quantum noise where only *bit flip* and *phase flip* errors can occur. We will see in the lecture tomorrow that this is without loss of generality, because Pauli errors $\{I, X, Z, Y = iXZ\}$ form an orthogonal basis for the Hilbert space of all 2×2 Hermitian matrices. This means, if a quantum code corrects for Pauli errors[1], it can correct arbitrary errors for free.

2.3 Fixing bit flip errors

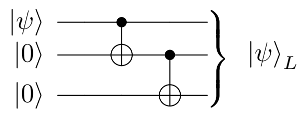
We now describe a 3-qubit quantum code that protects against bit flip errors. This was first investigated by Asher Peres in 1985.

Definition 2.5 (3-qubit bit flip code). *We define the code on the standard basis, and then extend via linearity.*

$$|0\rangle \to |000\rangle$$
$$|1\rangle \to |111\rangle$$
$$|\psi\rangle = a|0\rangle + b|1\rangle \to a|000\rangle + b|111\rangle = |\psi\rangle_L$$

[1]since Y error is just a phase flip error and a bit flip error, we actually only need to worry about X and Z errors

How do we implement this code? Since we can not clone quantum states, we can not go from $|0\rangle$ to $|000\rangle$. To implement this, we first add two ancilla qubits. To familarize us with quantum circuits, here is a circuit which implements this code:



The gate used above is a CNOT gate (Definition 1.5).

We encode $|\psi\rangle = a|0\rangle + b|1\rangle$, and send the qubits $|\psi\rangle_L = a|000\rangle + b|111\rangle$ through the quantum channel. Now, suppose there is a bit flip error, say on the second qubit.

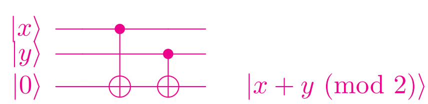
$$(I \otimes X \otimes I) \cdot |\psi\rangle_L = a|010\rangle + b|101\rangle = |\widetilde{\psi}\rangle_L$$

To detect a bit flip error, a natural strategy could be to measure the 3 qubits in the standard basis. This measurement gives string 010 with probability $|a|^2$, and string 101 with probability $|b|^2$. Using this, we can detect that a bit flip error occurred. However, in the process, we have collapsed our state $|\widetilde{\psi}\rangle_L$, and therefore, we can not recover the original state.

The issue is that we have measured too much. We should try to measure only "Has a bit flip error occurred, and on which qubits?" i.e. (1) Measure parity of first and second qubits, and (2) Measure parity of second and third qubits.

Both parities are 1 and 1, for both $|010\rangle$ and $|101\rangle$, so measuring it does not collapse the state. In other words, we perform the following circuit on $|\widetilde{\psi}\rangle_L$.

Scribe Task: Draw a circuit which computes the parity of 1st and 2nd qubit, and parity of 2nd and 3rd qubit. To begin, here is a circuit which computes parity of 1st and 2nd qubit.



The circuit acts on $|\widetilde{\psi}\rangle_L = a\,|010\rangle + b\,|101\rangle$ as follows:

$$
\begin{aligned}
|\widetilde{\psi}\rangle_L \otimes |00\rangle &= a\,|010\rangle \otimes |00\rangle + b\,|101\rangle \otimes |00\rangle \\
&\xrightarrow[\text{parity on 2 ancilla}]{\text{circuit computing}} a\,|010\rangle \otimes |11\rangle + b\,|101\rangle \otimes |11\rangle \\
&= |\widetilde{\psi}\rangle_L \otimes |11\rangle
\end{aligned}
$$

By measuring the last two registers in computational basis (which gives the measurement outcome 11), we can infer that the bit flip error occured on 2nd qubit. The measurement outcome 11 is called syndrome, and it gives the required information about the error, i.e. in this case, a bit flip occured on the second qubit. We can infer other errors from the syndrome as follow:

Syndrome	Error
00	no error
01	bit flip on 3rd qubit
10	bit flip on 1rd qubit
11	bit flip on 2rd qubit

In our case, we can fix the error by applying second bit flip.

$$(I \otimes X \otimes I) \cdot |\widetilde{\psi}\rangle_L = |\psi\rangle_L$$

Note that this code can not fix two bit flip errors. Moreover, it can not fix phase flip errors.

$$
\begin{aligned}
(Z \otimes I \otimes I) \cdot |\psi\rangle_L &= (Z \otimes I \otimes I)(a\,|000\rangle + b\,|111\rangle) \\
&= a\,|000\rangle - b\,|111\rangle
\end{aligned}
$$

We can not even detect this error because it is an encoding of state $a\,|0\rangle - b\,|1\rangle$.

Observation 2.6 (Measuring parity)**.** *Here is another interpretation for the parity measurement. We can measure "parity of 1st and 2nd qubit" and "parity of 2nd and 3rd qubit" by measurements of observables Z_1Z_2 and Z_2Z_3. Here the notation Z_iZ_j means Z gate applies on ith and jth qubit, and I on all remaining qubits.*

In our case, Z_1Z_2 has the spectral decomposition

$$Z_1Z_2 = (|00\rangle\langle 00| + |11\rangle\langle 11|) \otimes I - (|01\rangle\langle 01| + |10\rangle\langle 10|) \otimes I$$

Notice how the eigenvalues characterize the parity, and therefore, measuring observable Z_1Z_2 on $|\widetilde{\psi}\rangle_L = a\,|010\rangle + b\,|101\rangle$ does not change the state, and outputs -1 and identifies the syndrome.

2.4 Fixing phase errors

We now describe a code which corrects phase flip errors. The main observation to fix phase errors is that X and Z errors are switched by Hadamard transform

$$X = HZH$$

Definition 2.7 (3-qubit phase flip code)**.** *We define the code on the standard basis, and then extend via linearity.*

$$\begin{aligned} |0\rangle &\to |+++\rangle \\ |1\rangle &\to |---\rangle \\ |\psi\rangle = a|0\rangle + b|1\rangle &\to a|+++\rangle + b|---\rangle = |\psi\rangle_L \end{aligned}$$

Let's see what happens when there is a phase flip error, say on second qubit

$$Z_2 |\psi\rangle_L = a|+-+\rangle + b|-+-\rangle = |\widetilde{\psi}\rangle_L$$

Now, we can locate the error just like before, by first transforming to $|0\rangle, |1\rangle$ bases. The circuit acts on $|\widetilde{\psi}\rangle_L = a|+-+\rangle + b|-+-\rangle$ as follows:

$$\begin{aligned} |\widetilde{\psi}\rangle_L \otimes |00\rangle &= a|+-+\rangle \otimes |00\rangle + b|-+-\rangle \otimes |00\rangle \\ &\xrightarrow[\text{first 3 qubits}]{\text{Hadamard on}} a|010\rangle \otimes |00\rangle + b|101\rangle \otimes |00\rangle \\ &\xrightarrow[\text{parity on 2 ancilla}]{\text{circuit computing}} a|010\rangle \otimes |11\rangle + b|101\rangle \otimes |11\rangle \\ &\xrightarrow[\text{first 3 qubits}]{\text{Hadamard on}} |\widetilde{\psi}\rangle_L \otimes |11\rangle \end{aligned}$$

In other words, the syndrome measurement can be performed by measuring the observable $H^{\otimes 3} Z_1 Z_2 H^{\otimes 3} = X_1 X_2$ and $H^{\otimes 3} Z_2 Z_3 H^{\otimes 3} = X_2 X_3$. In our case, we can fix the error by applying phase flip to the second qubit.

2.5 9-qubit Shor code

We now discuss a simple code, Shor's 9-qubit code, which protects against arbitrary single qubit errors! This code is an example of code concatenation: a combination of 3-qubit bit flip code and 3-qubit phase flip code. We first encode with phase flip code: $|0\rangle \to |+++\rangle$ and $|1\rangle \to |---\rangle$, and then encode each of these qubits with bit flip code: $|+\rangle \to (|000\rangle + |111\rangle)/\sqrt{2}$ and $|-\rangle \to (|000\rangle - |111\rangle)/\sqrt{2}$.

Definition 2.8 (9-qubit Shor code)**.** *We define the code on the standard basis, and then extend via linearity.*

$$\begin{aligned} |0\rangle &\to \left(\frac{|000\rangle + |111\rangle}{\sqrt{2}}\right)^{\otimes 3} \stackrel{\text{def}}{=} |0\rangle_L \\ |1\rangle &\to \left(\frac{|000\rangle - |111\rangle}{\sqrt{2}}\right)^{\otimes 3} \stackrel{\text{def}}{=} |1\rangle_L \\ |\psi\rangle = a|0\rangle + b|1\rangle &\to a|0\rangle_L + b|1\rangle_L = |\psi\rangle_L \end{aligned}$$

Now, suppose a bit flip error occurs on the second qubit of the first block

$$\begin{aligned}
I \otimes X \otimes I^{\otimes 7} |\psi\rangle_L &= (I \otimes X \otimes I^{\otimes 7}) \cdot (a|0\rangle_L + b|1\rangle_L) \\
&= a\left(\frac{|010\rangle + |101\rangle}{\sqrt{2}}\right) \cdot \left(\frac{|000\rangle + |111\rangle}{\sqrt{2}}\right)^{\otimes 2} \\
&+ b\left(\frac{|010\rangle - |101\rangle}{\sqrt{2}}\right) \cdot \left(\frac{|000\rangle - |111\rangle}{\sqrt{2}}\right)^{\otimes 2} \\
&= |\widetilde{\psi}\rangle_L
\end{aligned}$$

We can now apply the circuit from bit flip detection on first 3 qubits to identify the error. In other words, we measure Z_1Z_2 and Z_2Z_3, and in our case, it will give measurement outcome 11.

Instead, if we had a phase flip error occur on the second qubit of the first block

$$\begin{aligned}
I \otimes Z \otimes I^{\otimes 7} |\psi\rangle_L &= (I \otimes X \otimes I^{\otimes 7}) \cdot (a|0\rangle_L + b|1\rangle_L) \\
&= a\left(\frac{|000\rangle - |111\rangle}{\sqrt{2}}\right) \cdot \left(\frac{|000\rangle + |111\rangle}{\sqrt{2}}\right)^{\otimes 2} \\
&+ b\left(\frac{|000\rangle + |111\rangle}{\sqrt{2}}\right) \cdot \left(\frac{|000\rangle - |111\rangle}{\sqrt{2}}\right)^{\otimes 2} \\
&= |\widetilde{\psi}\rangle_L
\end{aligned}$$

To identify the phase flip error, we measure $X_1X_2X_3X_4X_5X_6$ and $X_4X_5X_6X_7X_8X_9$. $X_1X_2X_3$ looks at the first block and returns $+1$ for $|000\rangle + |111\rangle$ and -1 for $|000\rangle - |111\rangle$. Similarly $X_4X_5X_6$ acts on the second block and therefore, $X_1X_2X_3X_4X_5X_6$ compares the sign of first two blocks and returns 1 if they are same and -1 otherwise.