Bilinear Classes: A Structural Framework for Provable Generalization in RL

Talk by: Gaurav Mahajan
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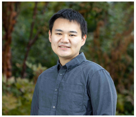
Joint work with:

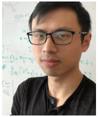


Simon Du

Sham Kakade

Jason Lee

Shachar Lovett



Wen Sun

Ruosong Wang

Progress of RL in practice

- Lots of recent empirical success.

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  - Growing theoretical work on assumptions which allow dealing with large state spaces.
  - Can we unify these assumptions?

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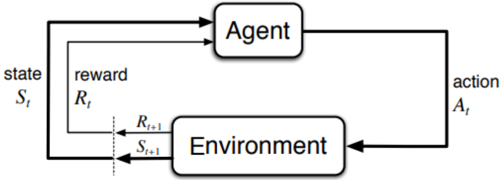
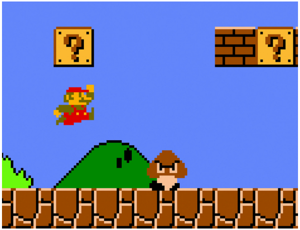
- Part I: Generalization in Reinforcement Learning
  Connections to Supervised Learning
- Part II: Unifying sufficient conditions
  Various model assumptions for generalization in RL
  Simple Algorithm and Short Proof

Markov Decision Processes: A Framework for RL





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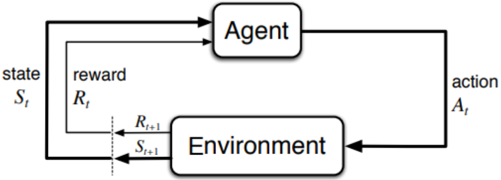

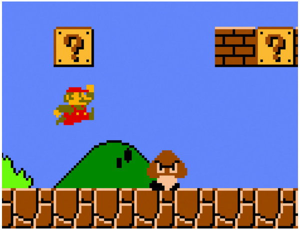

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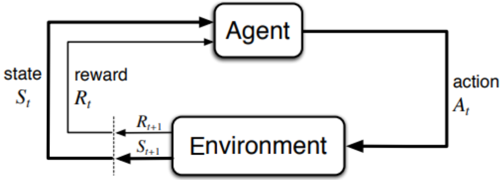

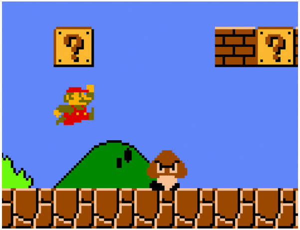

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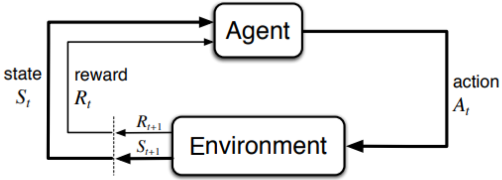

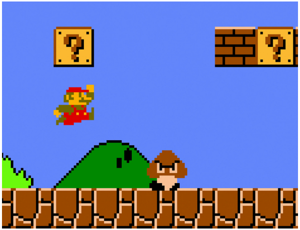

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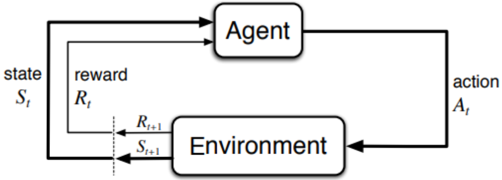

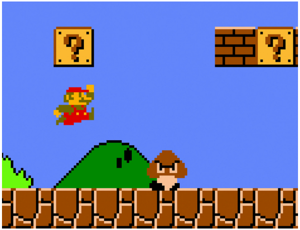

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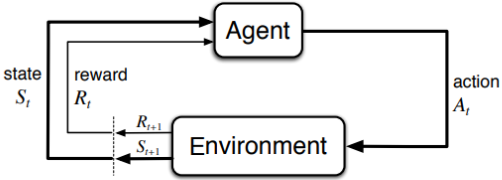

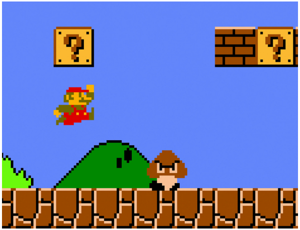

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Goal

Learn a policy $\pi : \mathcal{S} \rightarrow \mathcal{A}$ which maximizes $\mathbb{E}_\pi \left[\sum_{t=0}^{H-1} r_t\right]$.

Part I: Generalization from Supervised Learning to Reinforcement Learning

Generalization is possible in the IID supervised learning setting!!

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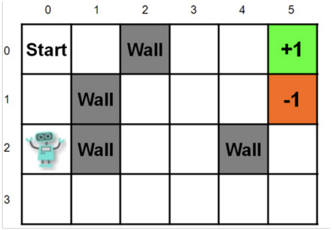
The key idea in SL: uniform convergence / data-reuse.
With a training set, we can simultaneously evaluate the loss of all hypotheses in our class!

Can we find an ϵ-opt policy with $\mathrm{poly}(\mathcal{S}, \mathcal{A}, H, 1/\epsilon)$ samples?



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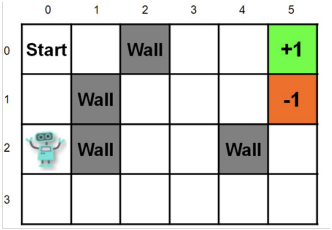

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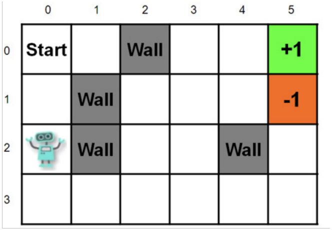

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- Key Idea: optimism + dynamic programming
- Add bonus for states which are not explored enough.

Provable Generalization in RL: Attempt I

Q1: Can we find an ϵ-opt policy with no $|\mathcal{S}|$ dependence?

Chess has $|S| \approx 10^{123}$
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- Can we avoid A^H dependence to find an ϵ-best-in-class policy?
  Without further assumptions, NO!!
  Proof: Consider a binary tree with 2^H policies and a sparse reward at a leaf node.

Provable Generalization in RL: Attempt II

Q2: Can we find an ϵ-opt policy with no $|\mathcal{S}|$, $|\mathcal{A}|$ dependence and $\mathrm{poly}(H, 1/\epsilon,$ "complexity measure"$)$?

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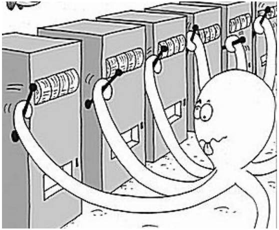
Q2: Can we find an ϵ-opt policy with no $|\mathcal{S}|$, $|\mathcal{A}|$ dependence and $\mathrm{poly}(H, 1/\epsilon,$ "complexity measure"$)$?

- With various stronger assumptions, YES!
  - Linear Bellman Completion: [Munos et al. '05, Zanette et al. '19]
    - Linear MDPs: [Wang & Yang '18]; [Jin et al.'19] (the transition matrix is low rank)
    - Generalized Linear Bellman Completion: [Wang et al. '2019]
  - FLAMBE / Feature Selection: [Agarwal et al. '20]
  - Linear Mixture MDPs: [Modi et al. '20, Ayoub et al. '20]
  - Block MDPs [Du et al. '19]
  - Factored MDPs [Sun et al. '19]
  - Kernelized Nonlinear Regulator [Kakade et al. '20]
    - Linear Quadratic Regulators (LQR): standard control theory model
  - And more...

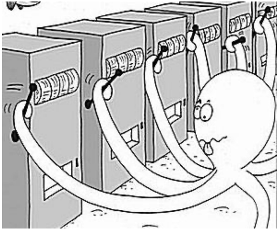
Part II: What are sufficient conditions for efficient RL?

Is there a common theme to prior settings?

Warm Up: Linear bandits ($H = 1$ RL problem) [Abe and Long, 1999]



- [Assumption 1] One step RL ($H = 1$): single state: s_0, large set of actions: $a \in \mathcal{A}$



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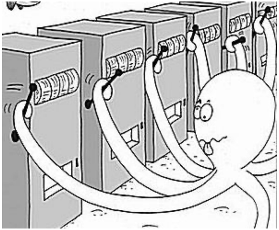
$$\mathbb{E}[r(s_0, a)] = \langle w^\star, \phi(s_0, a) \rangle$$



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Polynomial sample complexity is possible here [Auer et al. 2002; Dani et al. 2008]

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- Linear "value-based" Hypothesis class $\mathcal{F}$:
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$$\mathbb{E}_{\pi_w}[\ell(s_0, a, r, w')] = \Big\langle w' - w^\star,\ \mathbb{E}_{\pi_w}[\phi(s_0, a)] \Big\rangle$$

Essentially, we can use data collected under π_w to estimate the bilinear form for w'

- Hypothesis class: $\{f \in \mathcal{F}\}$
  with associated state action value $Q_f(s,a)$, (greedy) value $V_f(s)$ and (greedy) policy π_f
  - can be model-based or value-based class.

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Definition

A $(\mathcal{F}, \ell)$ forms an (implicit) Bilinear class if there exists $w_h : \mathcal{F} \to \mathbb{R}^d$ and $\Phi_h : \mathcal{F} \to \mathbb{R}^d$ for all timesteps $h \in [H]$:

BiLinear Classes: structural properties to enable generalization in RL

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$$\left| E_{\pi_f}\left[Q_f(s_h, a_h) - r(s_h, a_h) - V_f(s_{h+1})\right]\right| \leq \left|\langle w_h(f) - w_h(f^\star), \Phi_h(f)\rangle\right|$$

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- Data reuse: There exists loss function $\ell_f(s_h, a_h, r_h, s_{h+1}, g)$ such that the bilinear form for any hypothesis g is estimable when playing π_f

$$\left| E_{\pi_f}\left[\ell_f(r_h, s_h, a_h, s_{h+1}, g) \right] \right| = \left| \langle w_h(g) - w_h(f^\star), \Phi_h(f) \rangle \right|$$

Theorem 1: Structural Commonalities and Bilinear Classes

Theorem (Du, Kakade, Lee, Lovett, M., Sun, Wang '21)

The following models are bilinear classes for some bounded discrepancy function $\ell(\cdot)$

- *Linear Bellman Completion: [Munos et al. '05, Zanette et al. '19]*
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- *And more. . .*

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Theorem 1: Structural Commonalities and Bilinear Classes

Theorem (Du, Kakade, Lee, Lovett, M., Sun, Wang '21)

The following models are bilinear classes for some bounded discrepancy function $\ell(\cdot)$

- *Linear Bellman Completion: [Munos et al. '05, Zanette et al. '19]*
  - *Linear MDPs: [Wang & Yang '18]; [Jin et al.'19] (the transition matrix is low rank)*
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- *FLAMBE / Feature Selection: [Agarwal et al. '20]*
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- The framework easily leads to new models (see paper).

Algorithm 1: BiLin-UCB

Input number of iterations T, estimator function ℓ, batch size m, confidence radius R

Initialize discrepancy function $\sigma : \mathcal{F} \to \mathbb{R}$ as $\sigma^2(\cdot) = 0$

for *iteration* $t = 0, 1, \ldots, T-1$ **do**

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return: the best policy π_f found

Theorem (Du, Kakade, Lee, Lovett, M., Sun, Wang '21)

Assume $(\mathcal{F}, \ell)$ *is a bilinear class with* $\Phi_h(f) \in \mathbb{R}^d$*, bounded* ℓ *and the class is realizable, i.e.* $Q^\star \in \mathcal{F}$*. Using* $\frac{d^2}{\epsilon^2} \cdot poly(H) \cdot \log(|\mathcal{F}|) \cdot \log(1/\delta)$ *trajectories, the BiLin-UCB algorithm returns an* ϵ*-opt policy (with prob.* $1-\delta$*).*

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- The proof is "elementary" using the elliptical potential function.[Dani et al., '08]
- Extends to infinite dimensional problems using max info gain γ_T [Auer et al., '02; Srinivas et al., '10; Abbasi-Yadkori et al., '11]

Proof intuition

- The proof follows from this lemma about existence of high quality policy.

Lemma (Existence of high quality policy)

Suppose we run the algorithm for $T \approx d$ iterations. Then, there exists $t \in [T]$ such that the following is true for hypothesis f_t:

$$V^\star - V^{\pi_{f_t}}(s_0) \leq 2H\sqrt{d} \cdot \underbrace{H\sqrt{\frac{\log(|\mathcal{F}|)}{m}}}_{\text{SL generalization error of } \ell}$$

Proof intuition

- Bilinear regret assumption and Optimism give an upper bound for sub-optimality.

Lemma (Bilinear Regret Lemma)

The following holds for all $t \in [T]$ w.h.p.:

$$V^\star - V^{\pi_{f_t}}(s_0) \le \sum_{h=0}^{H-1} \left| \langle w_h(f_t) - w_h(f^\star), \Phi_h(f_t) \rangle \right| .$$

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$$V^\star(s_0) - V^{\pi_{f_t}}(s_0)$$
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$$\begin{aligned} & V^\star(s_0) - V^{\pi_{f_t}}(s_0) \\ & \leq V_{f_t}(s_0) - V^{\pi_{f_t}}(s_0) && \text{(optimism)} \\ & = \sum_{h=0}^{H-1} \mathbb{E}_{a_{0:h} \sim \pi_{f_t}} \left[Q_{f_t}(s_h, a_h) - r(s_h, a_h) - Q_{f_t}(s_{h+1}, a_{h+1}) \right] && \text{(telescoping sum)} \end{aligned}$$

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Proof of main lemma

- Bilinear regret assumption and Optimism give an upper bound on sub-optimality for all iterations t.

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- From Elliptical Potential Lemma, there exists $t \in [T]$ (for $T \approx d$) such that

$$\|\Phi_h(f_t)\|^2_{\Sigma_{t;h}^{-1}} = O(1) \quad \text{for all } h \in [H]$$

Note that for infinite dimensional spaces, we can use max info gain instead.

Lemma (Elliptical Potential Lemma; Dani et al., '08)

Consider any sequence of vectors $\{x_0, \ldots, x_{T-1}\}$ *where* $x_i \in \mathcal{V}$ *for some Hilbert space* $\mathcal{V}$*. Let* $\lambda \in \mathbb{R}^+$*. Denote* $\Sigma_0 = \lambda I$ *and* $\Sigma_t = \Sigma_0 + \sum_{i=0}^{t-1} x_i x_i^\top$*. We have that:*

$$\min_{i \in [T]} \ln\left(1 + \|x_i\|^2_{\Sigma_i^{-1}}\right) \leq \frac{1}{T} \ln \frac{\det\left(\Sigma_T\right)}{\det(\lambda I)}.$$

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- Proof: By definition of Σ_t and matrix determinant lemma, we have:

$$\ln\det(\Sigma_{t+1}) = \ln\det(\Sigma_t) + \ln\left(1 + \|x_t\|^2_{\Sigma_t^{-1}}\right).$$

- [Assumption 1] Linear $Q^\star$: There exists unknown $w^\star \in \mathbb{R}^d$ and known features $\phi : \mathcal{S} \times \mathcal{A} \to \mathbb{R}^d$ such that

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Special case II: Linear Bellman complete classes [Munos, 2005]

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Polynomial sample complexity is possible here [Zanette et al. 2020])

Special case II: Important structural property

Analogous structural property holds here:

- Bilinear Regret: on policy difference between claimed reward $\mathbb{E}[Q_w - V_w]$ and true reward $\mathbb{E}[r]$ satisfies a bilinear form

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Linear Function Approximation

Basic idea: approximate the $Q(s, a)$ values with linear basis functions $\phi_1(s, a), \ldots, \phi_d(s, a)$ (where $d \ll \#states, \#actions$).

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- C. Shannon. Programming a digital computer for playing chess. Philosophical Magazine, '50.
- Lots of work on this approach, e.g. TD-Gammon [Tesauro, '95], Atari [Mnih+ '13].

Special case III: Linear $Q^\star$

Theorem (Weisz, Amortila, Szepesvári '21)

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Theorem (Wang, Wang, Kakade '21)

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Can we get polynomial sample complexity
by also assuming linear $V^\star$?

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Special case IV: Linear Mixture Model classes [Modi et al., 2020b]

- [Assumption 1] Linear dynamics and rewards: There exists unknown $w^\star \in \mathbb{R}^d$ and known features $\phi : \mathcal{S} \times \mathcal{A} \times \mathcal{S} \to \mathbb{R}^d$, $\psi : \mathcal{S} \times \mathcal{A} \to \mathbb{R}^d$ such that

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Polynomial sample complexity is possible here [Modi et al., 2020; Ayoub et al., 2020])

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Special case V: Feature Selection for Low Rank MDPs

- [Assumption 1] Low rank MDP: There exists unknown features $\phi : \mathcal{S} \times \mathcal{A} \to \mathbb{R}^d$, $\psi : \mathcal{S} \to \mathbb{R}^d$ such that

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Polynomial sample complexity is possible here [Agarwal et al. 2020]

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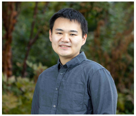
Thanks!

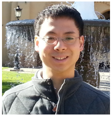
- A generalization theory in RL is possible!
  - linear bandit theory $\rightarrow$ RL theory (bilinear classes) is rich.
    - covers known cases and new cases
    - leads to simple algorithm and proof

- Open Questions
  - Computational - Statistical Tradeoff.
  - Agnostic Realizable Equivalence



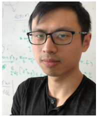
Simon Du

Sham Kakade



Jason Lee

Shachar Lovett



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Ruosong Wang